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Propagation of hydromagnetic planetary waves on a beta-plane through magnetic and velocity shear

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The propagation of hydromagnetic planetary waves in a rotating thin shell of fluid in regions of magnetic and velocity shears is studied using the β-plane approximation. In slowly varying shear the use of the WKBJ approximation makes it possible to construct the various types of ray trajectory that can occur and consequently the conditions that give rise to critical-latitude phenomena and trapping are deduced.

The opposite extreme to the WKBJ limit, namely reflexion and refraction of waves by a current–vortex sheet, is also analysed. In this case the conditions that lead to wave amplification (or over-reflexion) are investigated. Qualitatively, it is found that reflected Alfvén modes are amplified if the jump in the flow speed across the sheet lies between two speeds which are respectively greater and less than the sum of the Alfvén speeds on either side of the sheet. Also, Rossby waves incident upon a sufficiently strong easterly flow can suffer over-reflexion.

The general case of reflexion and refraction at a finite double (magnetic and velocity) shear layer is discussed. In analogy with the invariance of 'wave action' of gravity waves in a shear flow we construct a quantity $\mathscr{A}$ which is invariant except at critical latitudes, where it is discontinuous. By using the asymptotic solutions near these critical latitudes and by adopting the proper matching procedure for the solutions on either side of these latitudes it is possible to relate the two constant values of $\mathscr{A}$ on either side of each critical latitude. These general results are then applied to various profiles of shear flow and magnetic field so as to elucidate the manner in which an incident wave is reflected from and transmitted through a double layer.

1. Introduction

The propagation properties of hydromagnetic waves in a rotating fluid are relevant to many geophysical and astrophysical applications. Hide (1966) investigated the properties of these waves on a β-plane and discussed their relevance to the secular variation of the geomagnetic field. Hide & Jones (1972) made an extensive numerical study of the dispersion relationship derived by Hide (1966). They found that the interaction between the magnetic field and the rotation allows two types of wave motion. One type corresponds to a westward-propagating Rossby wave and is modified by magnetic effects, while the other corresponds to westward- and eastward-propagating Alfvén waves which, because of the effect of rotation, can propagate across magnetic lines of force. If the Alfvén speed is sufficiently large, the westward-propagating Rossby and Alfvén waves coalesce into one mode. An alternative way to view this is to note that a Rossby wave propagating into regions of increasing magnetic field strength can be converted into an Alfvén wave. Acheson (1973), in his study of hydromagnetic waves in a uniformly rotating fluid, has shown that the simultaneous action of angular velocity and magnetic field can give rise to other novel features, i.e. the valve effect.

The scatter of data on the magnetic fields and flow velocities in the interiors of celestial bodies (e.g. the earth) admits several theoretical interpretations. For example, estimates of the magnetic field in the earth's liquid core (which is predominantly zonal and cannot be measured on the earth's surface because of the weak. conductivity of the mantle) vary from 100 G (Hide 1966), for which good agreement with the geomagnetic secular variations is obtained, and $5G$ (Busse 1975), for which a successful dynamo model can be constructed. Also, the magnitude of the velocity varies from 4×10^{-4} m/s (Roberts & Soward 1972) to $0{\cdot}4\times10^{-2}$ m/s (Busse 1975). Thus flow speeds in the earth's liquid core may exceed Alfvén speeds, in which case critical-latitude absorption and emission (see § 6) may be relevant. The present work is mainly motivated by the need to elucidate the general properties of hydromagnetic waves in rotating (dissipationless) systems. However, certain simplifying assumptions like the β-plane approximation (which admittedly is justifiable only for motions in thin shells of fluid; Stewartson 1967) have been made in order to make the analysis tractable. It is our intention to relax this assumption in a future study.

In the next section we set up the equations governing the propagation of hydromagnetic planetary waves on a β-plane in the presence of a zonal flow and a 'toroidal' magnetic field which vary with latitude. The differential equation for the northward velocity perturbation is derived and its properties are analysed in the subsequent sections. In § 3 some general properties of the equations are discussed. In § 4 the results derived using the WKBJ approximation in § 3 are applied to some velocity–magnetic shear profiles. The various types of ray trajectory that can arise are constructed by using the geometrical properties of the locus of real wavenumbers (Lighthill 1967; McKenzie 1972).

In § 5 we examine reflexion and refraction of waves at a current–vortex sheet, the results of which may give a good approximation to the case where the latitudinal wavelength greatly exceeds the length scale of variation of the background state, i.e. the opposite extreme to slowly varying media. In particular we find that the wave can be amplified (i.e. the coefficient of reflexion exceeds unity), thereby extracting energy from the streaming motion. Since ordinary two-dimensional Rossby waves cannot be amplified at a vortex sheet and since a magnetic shear acting alone does not favour wave amplification (in the context of the present problem), the implication of this result is that wave amplification is due to the simultaneous action of the magnetic and velocity shear. In qualitative terms we find that eastward-propagating Alfvén waves can be amplified if the jump in the flow speed lies between two speeds which are respectively somewhat less and somewhat more than the sum of the Alfvén speeds on either side of the discontinuity. In this situation an incident eastward-propagating wave is transmitted as a westward-propagating Alfvén wave that has been 'blown' eastwards by the rapid westerly wind. Similar results apply to westward-propagating Alfvén waves. In addition it has been shown that Rossby waves can be amplified in a sufficiently strong easterly flow. In this case the incident Rossby wave is transmitted as an eastward-propagating Alfvén wave that has been 'blown' westwards by the easterly wind.

In § 6 we investigate some properties of reflexion and refraction of planetary waves by a finite shear layer in which both the magnetic field and the flow vary. This corresponds to a full wave treatment as opposed to on the one hand the WKBJ approximation (§ 4) and on the other the study of the layer as a discontinuity (§ 5). By using the wave invariant of the system (which is constructed in § 3) as a measure of the intensity of the wave it is possible to isolate certain novel features of the propagation of waves in a finite double shear layer. For example, it is shown that if a wave incident on a westerly wind increasing with height in the presence of a uniform magnetic field encounters *one* critical latitude (i.e. a latitude where the zonal phase speed of the wave matches the Alfvén speed) then it is over-reflected provided that the wave transmitted beyond the far end of the layer is evanescent. In the investigation of a current–vortex sheet in § 4 this situation corresponds to perfect reflexion. The reason for the discrepancy in the result for a current–vortex sheet, for which the influence of the critical latitudes within the sheet is ignored, is traced to the fact that the main cause of over-reflexion in this particular case is that the wave extracts energy from the background state *at the critical latitude*, i.e. wave amplification is due to critical-latitude emission. Other results are given in § 6.

In an appendix we derive the reflexion and transmission coefficients for a finite shear layer and also discuss briefly the stability properties of a current–vortex sheet.

2. The governing equations

The equations of motion, induction and continuity and Gauss' law for an incompressible, inviscid, perfectly conducting fluid rotating uniformly with angular velocity $\mathbf{\Omega}$ in the presence of a magnetic field $\mathbf{B}$ are

$$\frac{D\mathbf{u}}{Dt}+2\mathbf{\Omega}\wedge\mathbf{u} = -\frac{1}{\rho}\nabla\left(p+\frac{\mathbf{B}^2}{2\mu}\right)+\frac{1}{\mu\rho}(\mathbf{B}.\nabla)\,\mathbf{B}, \tag{2.1}$$

$$\partial\mathbf{B}/\partial t = \operatorname{curl}(\mathbf{u}\wedge\mathbf{B}), \tag{2.2}$$

$$\operatorname{div}\mathbf{u} = 0, \quad \operatorname{div}\mathbf{B} = 0, \tag{2.3}$$

in which $\mathbf{u}$ is the fluid velocity, p the pressure, ρ the density and μ the magnetic permeability. All quantities are measured in a frame of reference rotating with the fluid.

Let us develop the equations governing the propagation of hydromagnetic waves in a rotating, homogeneous, thin shell of fluid of radius r through regions of magnetic and velocity shear by making use of the β-plane approximation (Hide 1966; Hide & Jones 1972). We take local rectangular co-ordinates (x eastwards, y northwards, neglecting radial variations) and a basic state which consists of a zonal directed flow and magnetic field sheared latitudinally, i.e.

$$\mathbf{u}_0 = U(y)\,\hat{\mathbf{x}}, \quad \mathbf{B}_0 = B(y)\,\hat{\mathbf{x}}, \tag{2.4}$$

where $\hat{\mathbf{x}}$ is a unit vector in the x direction, and in which the total pressure $p_0 + \mathbf{B}_0^2/2\mu$ is given by

$$fU = -\frac{\partial}{\partial y}(p_0 + B_0^2/2\mu). \tag{2.5}$$

If $\mathbf{u} = (u, v)$ and $\mathbf{b} = (b_x, b_y)$ are the perturbations in the velocity and magnetic field, the linearized equations can be written as

$$\frac{Du}{Dt} + vU' - fv = -\frac{1}{\rho}\frac{\partial \Pi}{\partial x} + \frac{1}{\mu\rho}\left(B\frac{\partial b_x}{\partial x} + b_y B'\right), \tag{2.6}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial \Pi}{\partial y} + \frac{B}{\mu\rho}\frac{\partial b_y}{\partial x}, \tag{2.7}$$

$$Db_x/Dt + vB' - b_y U' = B\,\partial u/\partial x, \tag{2.8}$$

$$Db_y/Dt = B\,\partial v/\partial x, \tag{2.9}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} = 0, \tag{2.10}$$

where
$$\Pi = p + Bb_x/\mu, \quad D/Dt \equiv \partial/\partial t + U\,\partial/\partial x,$$

$$f = f_0 + \beta y, \quad \beta = 2\Omega\cos\theta_0/r, \quad f_0 = 2\Omega\sin\theta_0,$$

in which p is the perturbation in fluid pressure and θ_0 is the latitude. In (2.6)–(2.8) a prime denotes differentiation with respect to the argument. *This notation will be adopted throughout the paper.*

Assuming perturbations of the form $\exp i(\omega t - k_x x)$ and eliminating all variables in favour of v, the northward velocity, we obtain the following differential equation governing the latitudinal structure:

$$\phi''(y) + g(y)\,\phi(y) = 0, \tag{2.11}$$

where
$$v = \phi h = \phi\exp\left(-\int\frac{a}{2}dy\right) = \frac{\hat{\omega}\phi}{(\hat{\omega}^2 - k_x^2V^2)^{\frac{1}{2}}} \tag{2.12}$$

and
$$\left.\begin{aligned} a &= -2k_x^2V(\hat{\omega}V' + VU'k_x)/\hat{\omega}(\hat{\omega}^2 - k_x^2V^2), \\ \hat{\omega} &= \omega - k_xU, \quad V = B(\mu\rho)^{-\frac{1}{2}}, \end{aligned}\right\} \tag{2.13}$$

$$g(y) = k_x^2\left\{\hat{\omega}\frac{(\beta - U'')}{k_x(k_x^2V^2 - \hat{\omega}^2)} - \frac{VV'' + V'^2}{k_x^2V^2 - \hat{\omega}^2} - 1 + \frac{k_xV(k_xVV'^2 + 2\hat{\omega}V'U' + k_xVU'^2)}{(k_x^2V^2 - \hat{\omega}^2)^2}\right\}. \tag{2.14}$$

Here $\hat{\omega}$ is the Doppler-shifted frequency and V the Alfvén speed.

3. Some general results

It follows from (2.11)–(2.14) with appropriate boundary conditions [such as, for example, (6.4)] that the quantity

$$\mathscr{A} = \mathrm{Re}\,(-ik_x^{-2}\phi^*\,d\phi/dy) \tag{3.1}$$

(cf. Eltayeb 1977), where the asterisk denotes the complex conjugate and Re refers to the real part, is a constant except at critical latitudes [i.e. at regular singular points of (2.11)], where it jumps discontinuously from one constant value to another.

Now consider the energy flux density per unit mass, **F** say (as measured by a stationary observer), of an incompressible, infinitely conducting fluid described by a velocity **U**, pressure p and magnetic induction **B**:

$$\mathbf{F} = p\mathbf{U} + \tfrac{1}{2}\mathbf{U}^2\mathbf{U} - \mu^{-1}[\mathbf{B}^2\mathbf{U} - (\mathbf{B}\,.\,\mathbf{U})\,\mathbf{B}]. \tag{3.2}$$

The first term on the right-hand side represents the rate of working of the pressure force, the second is the flux of kinetic energy and the last two terms, involving **B**, represent the flux of electromagnetic energy, described by the Poynting vector $\mu^{-1}\mathbf{E}\wedge\mathbf{B}$, where **E** is the electric field, given by $\mathbf{E} = -\mathbf{U}\wedge\mathbf{B}$, for a perfectly conducting fluid (see, for example, Eskinazi 1967, § 10.6). The mean (denoted by an overbar) northward flux of energy associated with propagation of hydromagnetic waves through the basic state can be calculated from (3.2) by expanding about the basic state (2.4). Using (2.6)–(2.10) to describe the perturbations we find that the mean flux $\bar{F}_{\text{wave}}$ of wave energy to second order is given by

$$\bar{F}_{\text{wave}} = \overline{\Pi v} + U\overline{uv}(1 - k_x^2\,V^2/\hat{\omega}^2). \tag{3.3}$$

Part of the Poynting vector, $B_0 b_x v$, has been combined with pv to give the first term $\overline{\Pi v}$, which represents the rate of working of the 'total' perturbation pressure. The second term consists of the northward flux $\frac{1}{2}2\mathbf{U}\,.\,\mathbf{u}v$ of kinetic energy due to the interaction of the wave with the background and the remainder is the northward component of the Poynting vector, which reduces to $\mu_0^{-1}Ub_xb_y$. From (2.6)–(2.10) this second term can be written as $\overline{\Pi v}\;k_x\,U/\hat{\omega}$, so that

$$\bar{F}_{\text{wave}} = \overline{\Pi v}\frac{\omega}{\hat{\omega}} = \frac{c}{c-U}\,\overline{\Pi v}, \quad c = \omega/k_x.$$

We now use the relation

$$ik_x^2\Pi = (\hat{\omega} - k_x^2\,V^2/\hat{\omega})\,dv/dy + k_x(k_x^2\,V^2 - \hat{\omega}^2)\,U'v/\hat{\omega}^2 + kf\hat{v} \tag{3.4}$$

for the total pressure Π and use (2.12) and (3.1) to find that

$$\bar{F}_{\text{wave}} = c\mathscr{A}. \tag{3.5}$$

Thus the invariant $\mathscr{A}$ is closely linked with the conservation of the mean northward flux of wave energy. In the absence of angular velocity and magnetic field but in the presence of gravitational effects Eliassen & Palm (1960) have shown that the Reynolds stress is conserved except at critical levels. Booker & Bretherton (1967), in their study on gravity waves, located an invariant quantity which is closely linked with the Reynolds stress and called it 'wave action'. In the present study the invariant $\mathscr{A}$ can be shown to be proportional to the sum of the Reynolds and Maxwell stresses, which in turn can be related to the wave energy flux (measured in an inertial frame) by (3.5).

The mean rate M of northward transfer of zonal momentum, which is given by

$$M = \overline{\rho_0 uv},$$

can also be expressed in terms of $\mathscr{A}$:

$$M = k_x \hat{\omega}^2 \mathscr{A}/(\hat{\omega}^2 - k_x^2 V^2). \tag{3.6}$$

It then follows that M experiences an infinite jump across critical latitudes.

Next we shall derive an equation for the invariant (i.e. the wave-action density) in slowly varying media where the WKBJ approximation may be adopted. The results will be exploited in the next section to deduce conditions for critical-latitude behaviour and for trapping of waves in certain regions of magnetic–velocity shear.

Consider a basic state in which U and V vary on a large length scale ϵ^{-1}. Let us examine the evolution of this state at large times (i.e. of order ϵ^{-1}). Let

$$\mathbf{V}_0 = V(Y,T)\,\hat{\mathbf{x}}, \quad \mathbf{u}_0 = U(Y,T)\,\hat{\mathbf{x}}, \quad \Pi = \epsilon^{-1}\Pi_0(Y,T), \tag{3.7}$$

where

$$X = \epsilon x, \quad Y = \epsilon y, \quad T = \epsilon t \tag{3.8}$$

(cf. Grimshaw 1975). The mean-field equations then give [cf. (2.1)–(2.3)]

$$\epsilon \frac{\partial U}{\partial T}\,\hat{\mathbf{x}} + fU\hat{\mathbf{y}} = -\frac{\partial \Pi_0}{\partial Y}\,\hat{\mathbf{y}}, \tag{3.9}$$

$$\epsilon\, \partial V/\partial T = 0. \tag{3.10}$$

If $\hat{\mathbf{u}}$, $\hat{p}$ and $\hat{\mathbf{v}}$ are the perturbations in $\mathbf{u}_0$, Π_0 and $\mathbf{V}_0$ respectively and if

$$\{\hat{\mathbf{u}}, \hat{p}, \hat{\mathbf{v}}\} = \operatorname{Re}\{(\mathbf{u}, p, \mathbf{v}) \exp i\theta\}, \tag{3.11}$$

where

$$\theta = \epsilon^{-1}\,\Theta(X, Y, T), \tag{3.12}$$

then the perturbation equations can be written as

$$-i\hat{\omega}\mathbf{u} + f\hat{\mathbf{z}} \wedge \mathbf{u} + i\mathbf{k}p - i(\mathbf{V}.\mathbf{k})\,\mathbf{u} = \epsilon \mathbf{Q}, \tag{3.13}$$

$$-i\hat{\omega}\mathbf{v} - i(\mathbf{V}.\mathbf{k})\,\mathbf{u} = \epsilon \mathbf{R}, \tag{3.14}$$

$$i\mathbf{k}.\mathbf{u} = -\epsilon\nabla.\mathbf{u}, \quad i\mathbf{k}.\mathbf{v} = -\epsilon\nabla.\mathbf{v}, \tag{3.15), (3.16}$$

in which

$$\left.\begin{aligned} &\omega = -\Theta_T, \quad \mathbf{k} = \nabla\Theta = (k_x, k_y) = (\partial\Theta/\partial X, \partial\Theta/\partial Y),\\ &\mathbf{Q} = -\mathbf{u}_T - \nabla p + (\mathbf{u}.\nabla)\,\mathbf{V} - (\mathbf{u}.\nabla)\,\mathbf{U},\\ &\mathbf{R} = -\mathbf{u}_T + (\mathbf{u}.\nabla)\,\mathbf{U} - (\mathbf{u}.\nabla)\,\mathbf{V}. \end{aligned}\right\} \tag{3.17}$$

The next step is to expand the perturbation quantities in ϵ. Thus we set

$$\mathbf{u} = \mathbf{u}_0 + \epsilon \mathbf{u}_1 + \ldots, \tag{3.18}$$

with similar expressions for Π and $\mathbf{v}$. To zeroth order in ϵ we get (3.13)–(3.16) with the right-hand sides set to zero. This set of homogeneous equations for

$\mathbf{u}_0$, p_0 and $\mathbf{v}_0$ will have a non-trivial solution only if a consistency condition is satisfied. This gives the dispersion relation

$$k^2[\hat{\omega}^2-(\mathbf{V}.\mathbf{k})^2]+\beta\hat{\omega}k_x = 0 \tag{3.19}$$

for hydromagnetic Rossby waves. When first-order terms in ϵ are considered we obtain another set of equations. Here the left-hand sides are the same as in (3.13)–(3.16) but with $\mathbf{u}$, p and $\mathbf{v}$ replaced by $\mathbf{u}_1$, p_1 and $\mathbf{v}_1$ while the right-hand sides are $\mathbf{Q}_0$, $\mathbf{R}_0$, $-\nabla.\mathbf{u}_0$ and $-\nabla.\mathbf{v}_0$, where $\mathbf{Q}_0$ and $\mathbf{R}_0$ are the values of $\mathbf{Q}$ and $\mathbf{R}$ given by (3.17) with $\mathbf{u}$ and $\mathbf{v}$ replaced by $\mathbf{u}_0$ and $\mathbf{v}_0$ respectively. This set of inhomogeneous equations will have a solution for $\mathbf{u}_1$, $\mathbf{v}_1$ and p_1 only if the right side of the system is orthogonal to the solution of the adjoint homogeneous problem (i.e. orthogonal to $(\mathbf{u}^*, p^*, \mathbf{v}^*)$; see Grimshaw 1975). Thus

$$(\tfrac{1}{2}|\mathbf{u}_0|^2+\tfrac{1}{2}|\mathbf{v}_0|^2)_T+\nabla.(\mathrm{Re}\,(p_0\mathbf{u}_0^*)) = \mathrm{Re}\,\{\mathbf{v}_0^*(\mathbf{v}_0.\nabla)\,\mathbf{U}-\mathbf{u}_0^*.(\mathbf{u}_0.\nabla)\,\mathbf{U}\}. \tag{3.20}$$

If we now define a quantity $\mathscr{A}$ by

$$\mathscr{A} = \rho_0(\overline{uv}-\overline{b_x b_y}) \tag{3.21}$$

and carry out the calculations for all the terms in (3.20) on the same lines as in the study by Grimshaw (1975) we find that

$$\partial\mathscr{A}/\partial T+\partial(\mathscr{A}v_g)/\partial Y = 0, \tag{3.22}$$

where we have assumed for simplicity that $\mathbf{u}_0$, p_0 and $\mathbf{v}_0$, as well as k_x and ω, are independent of X although this restriction is not necessary, and

$$\mathbf{v}_g = (\partial\omega/\partial k_x, \partial\omega/k_y)$$

is the group velocity calculated from (3.19).

It is noteworthy that the conservation equation (3.22) shows that the energy density of the system (as measured by an observer moving with the basic flow) is

$$\begin{aligned} E &= \frac{\rho_0}{4}[|\mathbf{u}|^2+|\mathbf{v}|^2] = \frac{\rho_0[\hat{\omega}^2+(\mathbf{V}.\mathbf{k})^2]}{\hat{\omega}^2}|\mathbf{u}|^2 \\ &= -\frac{k^2[\hat{\omega}^2+(\mathbf{V}.\mathbf{k})^2]}{2k_y k_x[\omega^2-(\mathbf{V}.\mathbf{k})^2]}\mathscr{A} = +\frac{k^4}{2k_y k_x^2\hat{\omega}\beta}[\hat{\omega}^2+(\mathbf{V}.\mathbf{k})^2]\mathscr{A} \end{aligned} \tag{3.23}$$

and that the wave energy flux is related to $\mathscr{A}$ by (3.5). We may also note that, although (3.22) is identical to the wave-action equations of Bretherton & Garrett (1968) and of Grimshaw (1975), the relation between $\mathscr{A}$ (which is equivalent to the wave-action density) and E here is different.

The WKBJ approximation then yields the two equations (3.19) and (3.22). The former is a *local* dispersion relation and is used in § 4 below to examine the various types of ray trajectory that can arise in magnetic and velocity shear. The latter equation is usually known as the amplitude equation since it involves $|\mathbf{u}|^2$. For example, in the steady state (i.e. $\partial/\partial T = 0$) we have

$$\mathscr{A}v_g = \frac{\rho_0 k_y^2(\hat{\omega}^2-k^2V^2)}{k_x\hat{\omega}k^2(\hat{\omega}^2+k^2V^2)}|\mathbf{u}|^2 = \text{constant}, \tag{3.24}$$

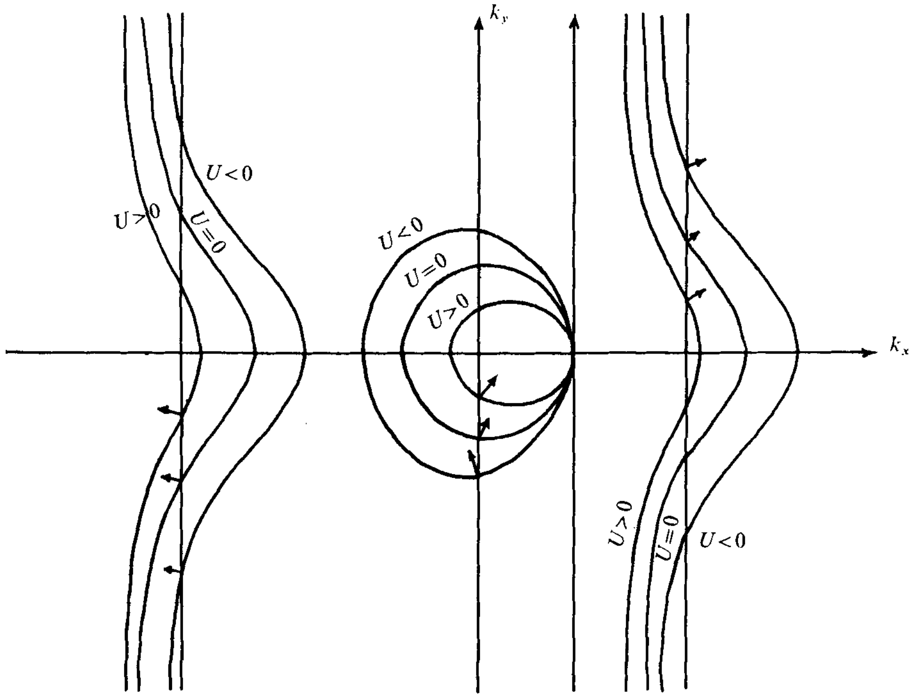


FIGURE 1. The locus of real wavenumbers at a fixed frequency ω for $U = 0$, $U > 0$ (westerly) and $U < 0$ (easterly). The closed loops represent Rossby waves modified by magnetic forces while the open branches represent Alfvén waves modified by Coriolis forces. The small arrows drawn normal to the locus of wavenumbers at points of intersection with lines k_x = constant indicate the direction of the ray (the group velocity).

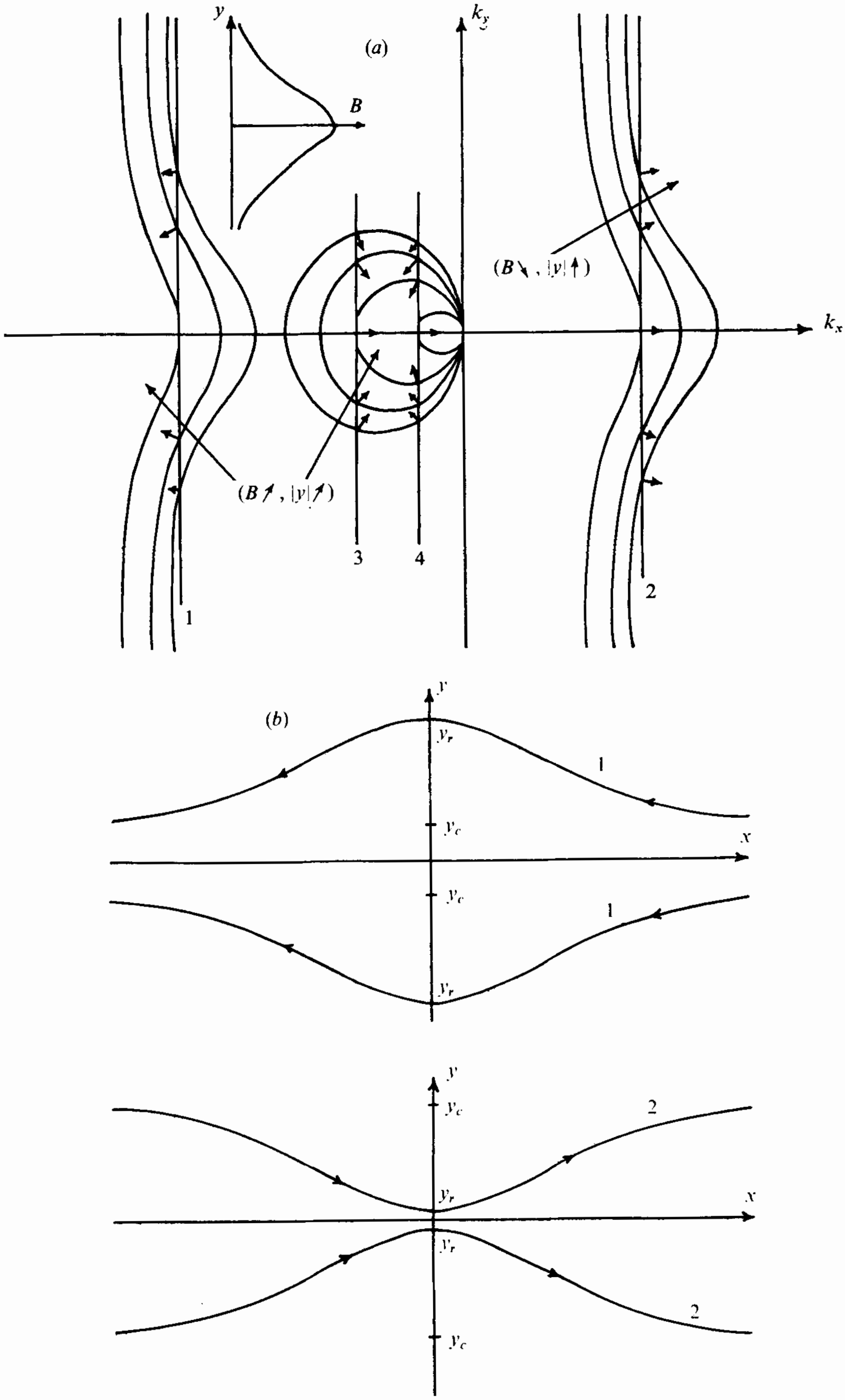
where the constant is determined by some particular values of k_y, ω, U, V and k_x. Thus the amplitude is a complicated function of these parameters. For a uniform basic state, however, a simple relation may be obtained:

$$|\mathbf{u}|^2 \simeq 1 + k_x^2/k_y^2. \tag{3.25}$$

Here the amplitude increases indefinitely as the reflexion points (i.e. $k_y = 0$) are approached while a minimum amplitude is attained at points where the wave is not propagating east–west (i.e. where $k_x = 0$).

The last general result concerns the singular points of (2.11)–(2.14), namely the points where $|a(y)| = \infty$ and where $|g(y)| = \infty$. These are, respectively, $\hat{\omega} = 0$ and $\hat{\omega} = \pm k_x V$. The former point does not lie on the locus of real wavenumbers, according to (3.19), and consequently does not correspond to a wave propagating through a uniform background state, i.e. the presence of this singularity is entirely due to the inclusion of variations in the background state and hence the wave invariant is continuous there (see Eltayeb 1977). The latter points, at which the zonal speed of the wave, as measured in a frame moving with the local flow speed, matches the Alfvén speed, correspond to critical latitudes which a ray (or wave packet), being neither reflected nor transmitted, approaches asymptotically. If a critical latitude occurs at $y = y_c$ then near $y = y_c$ equation (2.11) is approximated by

$$\phi'' + \phi/4(y-y_c)^2 = 0. \tag{3.26}$$



FIGURES 2 (*a*, *b*). For legend see next page.

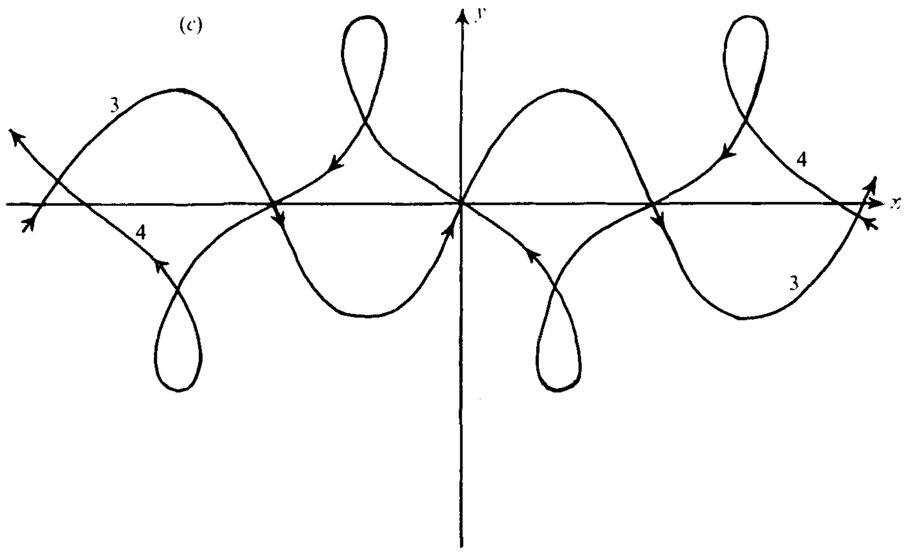


FIGURE 2. (a) Wave normal diagrams at successive latitudes for jet-like variation of B (or V) with latitude. The ray path associated with each type of wave can be constructed by following the direction of the arrows at each latitude for a given value of k_x. (The diagram has been drawn for $V(0) < V_c$ and $U < V_c$.) (b) 'Alfvén' ray trajectories exhibiting critical latitudes y_c at $k_x = \omega(U + V(y_c))$ and reflexion points y_r where $k_y^2 = 0$. The rays labelled 1 (2) correspond to westward (eastward) propagation. (c) 'Rossby' rays trapped around the centre of the magnetic jet. The ray labelled 3 (4) represents eastward (westward) energy propagation.

This equation is free of any parameters characterizing the strength of the shear of the zonal flow and magnetic field. (In fact it is exactly the same equation as that governing the propagation of gravity waves in a shear flow characterized by a Richardson number of $\frac{1}{4}$.) This feature contrasts with non-hydromagnetic Rossby waves, which, near $y = y_c$, are governed by the equation

$$d^2\phi/dz^2 + \phi/z = 0, \tag{3.27}$$

where

$$z = (y - y_c)(\beta - U'')/U', \tag{3.28}$$

which shows that the behaviour of ordinary Rossby waves near a critical latitude is characterized by a length scale

$$L = |U'(y_c)/(\beta - U''(y_c))|. \tag{3.29}$$

We also note that the singularity is changed from the $(y - y_c)^{-2}$ type which prevails in the presence of the magnetic field to a $(y - y_c)^{-1}$ type in its absence.

4. Ray trajectories in magnetic–velocity shear

In this section we examine the various types of ray trajectory that arise in slowly varying magnetic and velocity shear. It is well known (see, for example, Longuet-Higgins 1965; Lighthill 1967) that the frequency ω and longitudinal wavenumber k_x are conserved along a ray path in the y, x plane. Thus, by noting

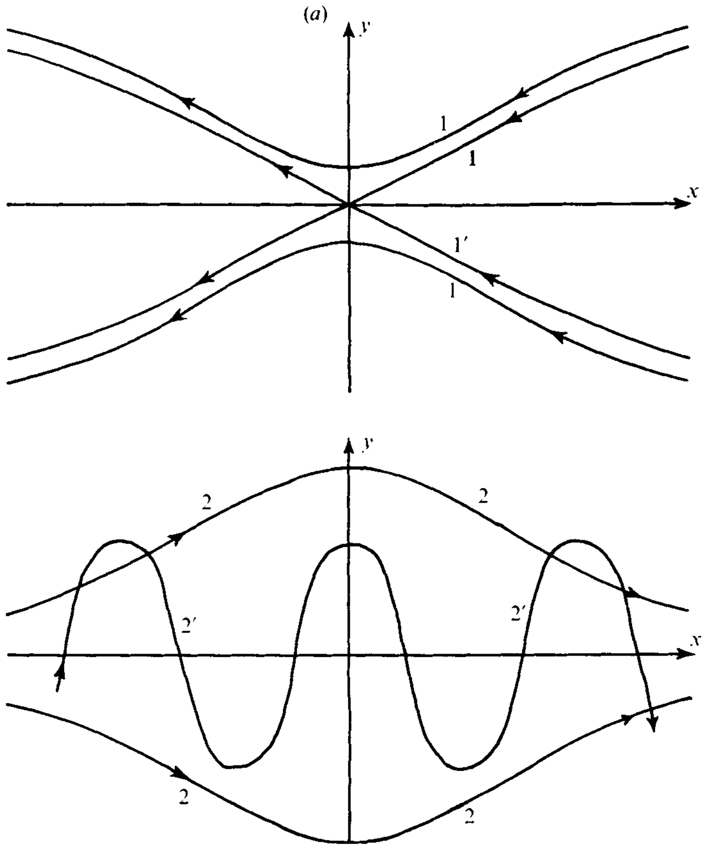


FIGURE 3(*a*). For legend see next page.

that the ray direction (i.e. the direction of the group velocity) is normal to the locus of real wavenumbers, the ray path can be constructed by finding the shapes of the loci of real wavenumbers at successive values of y, for any given variation of the zonal flow speed and magnetic field with y. (For other applications of this geometrical construction see, for example, Lighthill 1967; McKenzie 1973.) In figure 1 we have sketched the wavenumber curves for $U = 0$, $U > 0$ (i.e. westerly wind) and $U < 0$ (easterly wind). It will be observed that, for a given ω and k_x, $|k_y|$ decreases for U positive and increasing, whereas for U negative and decreasing $|k_y|$ increases.

By using the above-mentioned geometrical construction we have inspected the various types of ray trajectory that are possible in two different magnetic shear profiles. One profile is a symmetric jet-like variation in the Alfvén speed (or magnetic field) with latitude and the other is an antisymmetric east–west shear in which the magnetic field is zero at $y = 0$.

Figure 2 illustrates the four different types of ray trajectory that can arise with a jet-like variation of the magnetic field. We have assumed that $|U| < V$ and also that $V < V_0$ at $y = 0$. Here V_0 is the value of V when the wave normal curves of the Rossby wave and the westward-propagating Alfvén wave coalesce

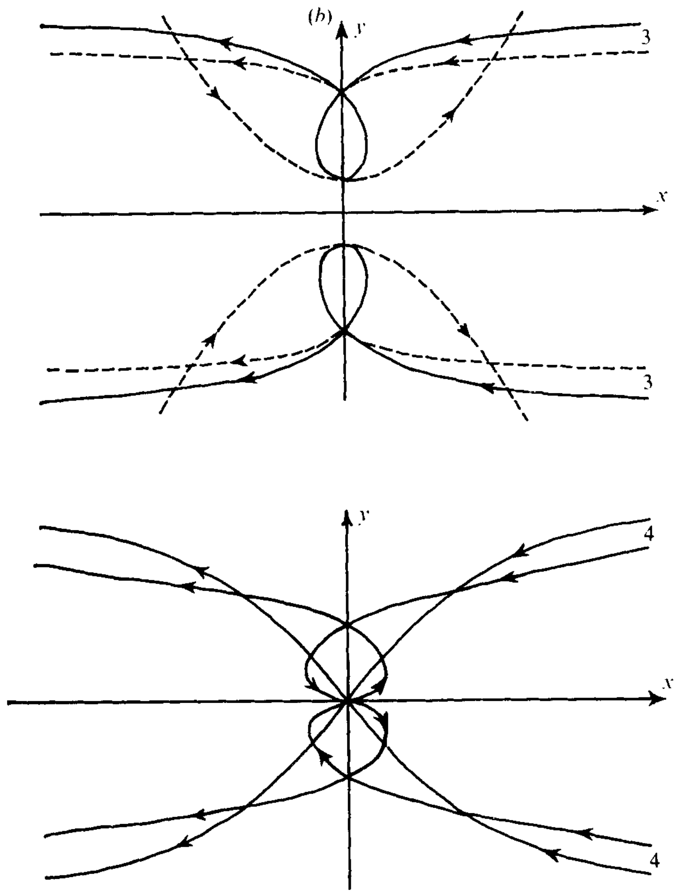


FIGURE 3. Ray trajectories for an east–west magnetic shear. These can be constructed by using the wave normal curves of figure 2(*a*), in which the direction of B increasing is reversed. (*a*) 'Alfvén' ray trajectories exhibiting critical latitudes. Rays 1 correspond to westward propagation and rays 2 to eastward propagation, for which it will be observed that trapping occurs if $k_x > \omega/(V(0)+U)$. (*b*) Modified Rossby ray trajectories. Rays 3 are confined to one side whereas rays 4 penetrate from south to north and vice versa. Critical levels for both types occur if the ultimate magnetic field strength is sufficiently large (i.e. if $k_x < \omega/(-V(\infty)+U)$).

to form one curve. (V_0 is a complicated function of U, ω and k_x, and has been calculated numerically by Hide & Jones 1972). Rays 1 and 2 correspond, respectively, to waves propagating westwards and eastwards in the Alfvén mode modified by the β-effect. These rays exhibit critical latitudes at

$$k_x = \omega/(U \mp V(y_c))$$

and are reflected from latitudes where $k_y = 0$. The sources of the rays can be any point on the ray path since integration of the equation for the ray path introduces an arbitrary constant which in any particular case is determined from the location of the source of the ray. For convenience we have chosen the origin in each case in such a way that the rays are symmetric about the y axis. The corresponding rays for $y < 0$ are obtained by reflecting rays 1 and 2 about the x axis.

Rays 3 and 4 correspond to westward phase propagation of Rossby waves modified by the magnetic field. These rays are trapped around the centre of the magnetic jet, again being reflected at latitudes for which $k_y = 0$. Ray 4 carries energy eastwards although its phase speed is westward.

Figure 3 illustrates the different types of ray trajectory that can arise in an antisymmetric magnetic shear. (Note that, since the dispersion relationship (12) is even in V (or B), those trajectories are also applicable to a symmetric shear in which $V = 0$ at $y = 0$.) Broadly speaking we see that there are three types of trajectory: (i) those that penetrate right through the shear from south to north (rays 1′ and 4) along with their reflexions about the x axis, which describe north-to-south penetration; (ii) those that are confined to one side or the other (rays 2 and 3); (iii) those that are trapped about the centre of the magnetic shear (ray 2′).

The various types of ray path that are possible in different forms of zonal flow with latitudinal shear have been discussed by Mekki & McKenzie (1977). The presence of the magnetic field introduces two more modes (which are essentially Alfvén waves modified by the Coriolis force) that are capable of exhibiting critical latitudes. Here we shall not discuss in detail the effect of zonal-flow shear on these modes but simply mention that such modes will exhibit ray paths of types 3 and 3′ in figure 2 and of types 3 and 4 in figure 3(*b*).

5. Reflexion and refraction of waves at a current–vortex sheet

In marked contrast to the previous section, where we discussed the propagation of waves in slowly varying media, we now consider the propagation of waves across discontinuities in the basic state. The results of the discussion may apply, asymptotically, to the case where the latitudinal wavelength $2\pi/k_y$ is very much greater than the length scale of variation of the basic state. (See Eltayeb & McKenzie (1975) for a formal treatment of such a limit for gravity waves incident upon a shear layer.)

We consider two uniform basic states separated by a current–vortex sheet located in the plane $y = 0$. The undisturbed flow velocities $U_1\hat{\mathbf{x}}$ and $U_3\hat{\mathbf{x}}$ and magnetic fields $B_1\hat{\mathbf{x}}$ and $B_3\hat{\mathbf{x}}$ are tangential to the sheet, where the subscripts 1 and 3 refer respectively to the regions $y < 0$ and $y > 0$. (The suffixes 1 and 3 are used here to facilitate comparison with the study of a finite shear layer in § 6.) A wave incident on the sheet from $y < 0$ gives rise to a reflected wave, a transmitted wave and a distortion of the sheet. Application of the boundary conditions (namely continuity of displacement and total pressure balance) at the distorted sheet determines the amplitudes of the reflected and transmitted waves.

In $y \lessgtr 0$ the northward velocity may be written in the forms

$$\left.\begin{aligned} v_1 &= \exp(ik_{y1}y) + R\exp(-ik_{y1}y), \quad && y < 0,\\ v_3 &= T\exp(ik_{y3}y), && y > 0,\end{aligned}\right\} \tag{5.1}$$

where $|R|$ and $|T|$ are respectively the reflexion and transmission coefficients. We have assumed perturbations of the form $\exp i(\omega t - k_x x)$ so that the k_{yi} $(i = 1, 3)$ satisfy the dispersion relationships

$$k_{yi}^2 + k_x^2 + \beta\hat{\omega}_i k_x/(\hat{\omega}_i^2 - k_x^2 V_i^2) = 0, \quad \hat{\omega}_i = \omega - k_x U_i. \tag{5.2}$$

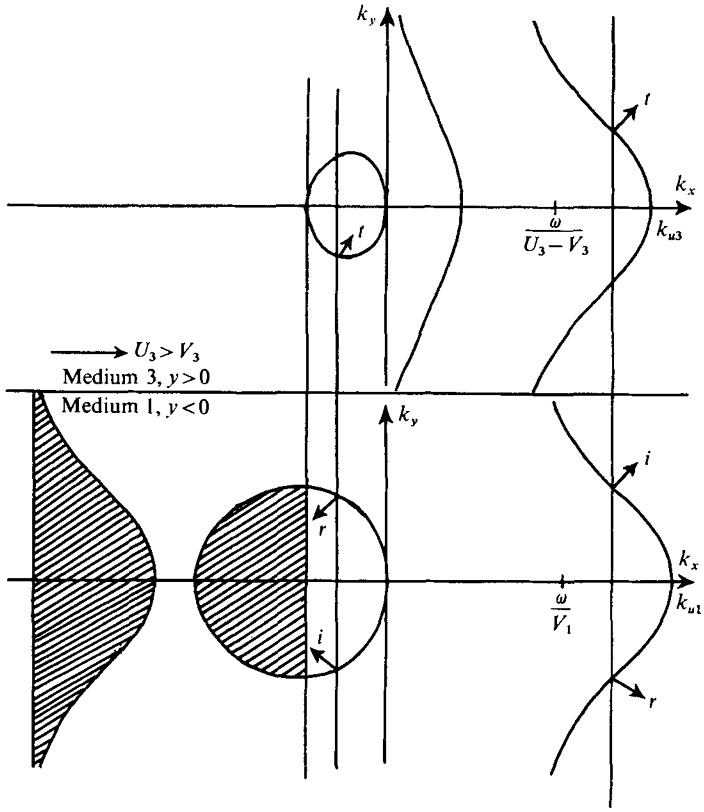


FIGURE 4. The wave normal diagrams on either side of a current–vortex sheet for flow speeds giving rise to wave amplification. Snell's laws of reflexion and refraction are illustrated geometrically by conserving k_x across the discontinuity. The arrows labelled i, r and t indicate the ray directions of the incident, reflected and transmitted waves. The shaded areas of the Rossby and Alfvén modes for $y < 0$ correspond to total reflexion.

The laws of reflexion and refraction follow from the continuity of ω and k_x across the discontinuity (see figure 4). The normal component k_y of the wavenumber on either side satisfies (3.19) and its sign must be chosen in such a way that the energy flux of the incident wave is directed towards the interface whereas the transmitted wave's energy flux must diverge from the interface. If $k_{y3}^2 < 0$, the sign is chosen to ensure that the amplitude of the transmitted wave decays in the region $y > 0$. In this case, in which k_{y3} is purely imaginary, (5.8) below shows that total reflexion (i.e. $|R| = 1$) occurs.

When the distortion of the sheet is written in the form

$$y = \eta \exp i(\omega t - k_x x), \tag{5.3}$$

the kinematic boundary condition, namely that the sheet is a streamline common to both flows, becomes, on linearization,

$$v_1 - i\hat{\omega}_1 \eta = v_3 - i\hat{\omega}_3 \eta = 0, \tag{5.4}$$

which immediately gives

$$v_1/\hat{\omega}_1 = v_3/\hat{\omega}_3 \quad \text{at} \quad y = 0. \tag{5.5}$$

On writing down the dynamic boundary condition that the total perturbation pressure be continuous we must remember that the linearized pressure perturbation consists of two parts, one of which is associated with the waves (i.e. Π) and the other of which is due to evaluating the basic state pressure Π_0 at the perturbed boundary (i.e. $\eta\,\partial\Pi_0/\partial y$). Thus

$$\Pi_1 - fU_1\eta = \Pi_3 - fU_3\eta, \tag{5.6}$$

where we have used (2.5). We now use the expression (3.4) for Π and (5.4) to eliminate η to find that

$$(\hat{\omega}_1^2 - k_x^2 V_1^2)\, v_1'/\hat{\omega}_1 = (\hat{\omega}_3^2 - k_x^2 V_3^2)\, v_3'/\hat{\omega}_3 \quad \text{at} \quad y = 0, \tag{5.7}$$

where a prime denotes differentiation with respect to y.

Substituting (5.1) into (5.5) and (5.7) we find the following expressions for R and T:

$$R = \frac{\hat{\omega}_1 k_{y1}(k_x^2 + k_{y3}^2) - \hat{\omega}_3 k_{y3}(k_x^2 + k_{y1}^2)}{\hat{\omega}_1 k_{y1}(k_x^2 + k_{y3}^2) + \hat{\omega}_3 k_{y3}(k_x^2 + k_{y1}^2)}, \tag{5.8}$$

$$T = (\hat{\omega}_3/\hat{\omega}_1)(1 + R), \tag{5.9}$$

in which we have made use of the dispersion relationship (3.9) in $y \lessgtr 0$.

Equation (5.8) shows that $|R| > 1$ if

$$\hat{\omega}_1 \hat{\omega}_3 < 0, \tag{5.10}$$

since k_{y1} and k_{y3} are always of the same sign for waves carrying energy northwards. If this condition is interpreted in terms of wave energy densities (see, for example, McKenzie 1970) it implies that wave amplification can arise in the present problem only if the energy density of the incident wave is of opposite sign to the energy density of the transmitted wave (but see Eltayeb 1977). Condition (5.10) can be written in terms of the magnitude of the jump in the flow speed across the sheet required to render wave amplification possible. Without any loss of generality we take medium 1 to be at rest so that $\hat{\omega}_1 = \omega$ (> 0 for definiteness); then we find that $\hat{\omega}_3$ can be negative only if

$$|U_3| > V_3, \tag{5.11}$$

i.e. if the jump in the flow speed exceeds the Alfvén speed in medium 3. However, this condition is not sufficient since in deriving it we have implicitly assumed that the transmitted perturbation is a propagating wave, i.e. that k_{y3} is real. Figure 4 illustrates geometrically the circumstances in which (5.10) is satisfied with k_{y3} real for westerly winds. Thus the necessary and sufficient conditions for hydromagnetic planetary waves to be amplified in a westerly shear may be written as

$$k_{u1} > \omega/(U_3 - V_3), \quad k_{u3} > \omega/V_1, \tag{5.12}$$

where k_{ui} ($i = 1, 3$) is the largest root of the cubic

$$(U_i^2 - V_i^2)\,[k_x - \omega/(U_i + V_i)]\,[k_x - \omega/(U_i - V_i)] + \beta(\omega - k_x U_i)/k_x = 0. \tag{5.13}$$

In qualitative terms conditions (5.12) imply that wave amplification is possible if the jump in the zonal flow speed lies between speeds which are respectively less and greater than the sum of the Alfvén speeds. This ensures that an incident positive-energy Alfvén wave propagating eastwards is transmitted as a negative-energy ($\hat{\omega}_3 < 0$) Alfvén wave propagating westwards (relative to medium 3) that is blown eastwards by the rapid westerly zonal flow.

We next consider medium 1 to be at rest and medium 3 to be flowing westwards (i.e. $U_3 < 0$) at a speed exceeding the Alfvén speed in medium 3 (i.e. $|U_3| > V_3$). If we assume that $V_1 < V_0$ so that the Alfvén and Rossby branches of the locus of wavenumbers are separate, we find that the following conditions are sufficient to ensure wave amplification:

$$-k_{ui} < \omega/(U_3 - V_3), \quad -k_{u3} < \omega/V_1, \tag{5.14}$$

where k_{ut} is the largest root k_x (in magnitude) of the cubic

$$(U_i^2 - V_i^2)\,[k_x + \omega/(|U_i| - V_i)]\,[k_x + \omega/(|U_i| + V_i)] + \beta(\omega - U_i k_x)/k_x = 0. \tag{5.15}$$

If these conditions are satisfied an incident positive-energy westward-propagating Alfvén wave is transmitted as a negative-energy eastward-propagating Alfvén wave that is 'blown westwards' by the easterly wind. Qualitatively conditions (5.14) imply that the jump in the flow speed lies between two speeds which are respectively less and greater than the sum of the Alfvén speeds. At still larger flow speeds we find that the reflected Rossby waves can be amplified with the transmitted wave being a negative-energy Alfvén wave blown westwards by the flow. In this case the condition for wave amplification is simply

$$|U_3| > V_3 + W_1, \tag{5.16}$$

where W_1 is the largest negative root of the cubic

$$W^2 - V_1^2 + W^3\beta/\omega^2 = 0. \tag{5.17}$$

If $V_1 > V_0$, so that the westward-propagating Alfvén and Rossby waves form a single branch of the locus of wavenumbers, the condition for wave amplification is

$$k_{u3} > -\omega/V_1, \tag{5.18}$$

where k_{u3} is the largest root of the cubic (5.15).

The conditions derived for wave amplification cannot be satisfied in the absence of the magnetic field, as can be seen from inspection of the wave normal curves (see also Mekki & McKenzie 1977). It can also be seen that wave amplification is not possible in the absence of the shear flow. It may therefore be concluded that wave amplification is facilitated by the simultaneous action of the magnetic field and shear flow.

6. Reflexion from a finite shear layer (critical latitudes)

Consider a finite shear layer of thickness L. Suppose that the magnetic field B and the velocity U are both parallel to the x axis, so that

$$V, U = \left\{\begin{array}{lll} V_1, 0 & \text{for} & y \leqslant 0, \\ V(y), U(y) & \text{for} & 0 \leqslant y \leqslant L, \\ V_3, U_3 & \text{for} & L \leqslant y. \end{array}\right\} \tag{6.1}$$

We shall number the regions $y \leqslant 0, 0 \leqslant y \leqslant L$ and $y \geqslant L$ as 1, 2 and 3 respectively. It is assumed that $V(y)$ and $U(y)$ are continuous at $y = 0$, L, so that $V(0) = V_1$, $U(0) = 0$, $V(L) = V_3$ and $U(L) = U_3$, where V_1, V_3 and U_3 are constant.

Our objective is to discuss reflexion from and transmission through the shear layer. An incident wave gives rise to a reflected wave of amplitude $|R|$ in region 1 and a transmitted wave of amplitude $|T|$ in region 3. The solutions in regions 1 and 3 are then given by (5.1) and (5.2). (Note that in a uniform medium $h = 1$ and $v = \phi$.) In region 2 the solution naturally depends on the particular choice of U and V. However, it is our purpose to establish some general results and to this end we may write

$$\phi = AW_1(y) + BW_2(y), \tag{6.2}$$

where W_1 and W_2 are the two independent solutions of (2.11). If the solutions W_1 and W_2 are known, the amplitudes R, T, A and B can be computed from the relations obtained by applying the boundary conditions at $y = 0, L$. The relevant boundary conditions here, namely continuity of both the normal component of velocity v and the perturbation total pressure Π, can be written in terms of ϕ as

$$[h\phi] = [\hat{\omega}(\hat{\omega}^2 - k_x^2 V^2)(h'\phi + h\phi') + h\phi U'(\hat{\omega}^2 - k_x^2 V^2) - fk_x \hat{\omega} h\phi] = 0 \text{ at } y = 0, L, \tag{6.3}$$

where the square brackets denote the jump. If we assume that U' and V' (where V is the Alfvén speed) are continuous at $y = 0$, L, conditions (6.3) reduce to

$$[\phi] = [\phi'] = 0. \tag{6.4}$$

The derivation of the expressions for R, T, A and B is relegated to the appendix.

We shall now establish some general results using the invariant of the system. Evaluation of $\mathscr{A}$ in regions 1 and 3 gives, respectively,

$$\mathscr{A}_s = (k_{y1}/k_x^2)(1 - |R|^2), \quad \mathscr{A}_n = (k_{y3}/k_x^2)|T|^2, \tag{6.5}$$

where the suffixes s and n refer to 'south' and 'north' of the shear layer. If there is no critical latitude within the layer then $\mathscr{A}$ takes the same value *everywhere* and thus

$$|R|^2 = 1 - (k_{y3}/k_{y1})|T|^2. \tag{6.6}$$

Now an inspection of the group velocity for the dispersion relationship (3.19) shows that waves propagating energy northwards are characterized by $k_{y1} k_{y3} > 0$. Hence $|R| < 1$ and *wave amplification* (or over-reflexion) *is not possible in the absence of a critical latitude whatever the behaviour of* $U(y)$ *and* $V(y)$ *in the layer.*

When a critical latitude exists within the layer and $\mathscr{A}$ is discontinuous there, progress can be made if we can solve (2.11) near the critical latitude $y = y_c$ and determine the correct matching for the solutions on either side of $y = y_c$. Now near a critical latitude (i.e. where $\hat{\omega}^2 = k_x^2 V^2$) (2.11) reduces, to the leading order, to

$$\phi'' + \phi/4(y - y_c)^2 = 0, \tag{6.7}$$

so that we may write

$$\phi = (y - y_c)^{\frac{1}{2}}[A + B\log(y - y_c)]. \tag{6.8}$$

This solution has a branch point at the critical latitude $y = y_c$. The matching procedure at this point can be determined either by considering an initial-value problem or by allowing ω and k_y to be complex with small imaginary parts. In the latter approach we impose the condition that the solution decays as $y \to \infty$ for fixed t but grows with time for fixed y, so that energy is propagated northwards (Miles 1961; Booker & Bretherton 1967). Thus for an exponential dependence $\exp\{i(\omega t - k_x x - k_y y)\}$, where $\omega = \omega_r + i\omega_i$ and $k_y = k_{yr} + ik_{yi}$, we must take $k_{yi} < 0$ and $\omega_i < 0$. With these values of ω and k_y, (6.8) becomes

$$\phi = \left[y - y_c - \frac{i\omega_i}{k_x(U_c' \pm V_c')}\right]^{\frac{1}{2}} \left\{A + B \log\left[y - y_c - \frac{i\omega_i}{k_x(U_c' \pm V_c')}\right]\right\}, \tag{6.9}$$

where a plus (minus) sign refers to the critical latitude where $k_x = \omega/(U_c \pm V_c)$ and U_c' and V_c' denote $U'(y_c)$ and $V_c'(y_c)$. Now the argument of $y - y_c - i\omega_i/k_x(U_c' \pm V_c')$ varies from zero for large positive values of $y - y_c$ to $+\pi$ or $-\pi$ for negative values of $y - y_c$ depending on whether $k_x(U \pm_c' V_c')$ is positive or negative. Accordingly the appropriate solutions on either side of the critical latitude can be written as

$$\phi = \begin{cases} |y-y_c|^{\frac{1}{2}}(A + B\log|y-y_c|) & \text{for} \quad y < y_c \\ +i|y-y_c|^{\frac{1}{2}}\{A + B(\log|y-y_c| + i\pi)\} & \text{for} \quad y > y_c \end{cases} \tag{6.10}$$

for $k_x(U_c' \pm V_c') < 0$ and

$$\phi = \begin{cases} |y-y_c|^{\frac{1}{2}}(A + B\log|y-y_c|) & \text{for} \quad y < y_c \\ -i\,|y-y_c|^{\frac{1}{2}}\{A + B(\log|y-y_c| - i\pi)\} & \text{for} \quad y > y_c \end{cases} \tag{6.11}$$

for $k_x(V_c' \pm V_c') > 0$.

The invariant $\mathscr{A}$ can now be evaluated at two points near but on opposite sides of the critical latitude. The calculations yield

$$\mathscr{A}_s = \operatorname{Re}(-iA^* B/k_x^2), \quad \mathscr{A}_n = \mathscr{A}_s \pm \pi|B|^2/k_x^2, \tag{6.12}$$

where the upper and lower signs refer to (6.10) and (6.11) respectively. Thus the invariant $\mathscr{A}$ is altered by an additive quantity in contrast to the multiplicative factor in the case of the wave action of gravity waves (Eltayeb & McKenzie 1975).

Use of (6.5) and (6.12) gives

$$|R|^2 = 1 - (k_{y3}/k_{y1})\,|T|^2 + (\pi\,|B|^2/|k_{y1}|)\operatorname{sgn}(U_c' \pm V_c'). \tag{6.13}$$

Apart from the critical latitudes there are two other latitudes which deserve a special comment before we discuss the four examples below. One type of latitude, $y = y_0$, say, occurs where the flow speed matches the zonal phase speed of the wave. $\mathscr{A}$ is continuous at $y = y_0$ since (2.11) is regular there, but a wave crossing this latitude changes the sign of its energy density since $\hat{\omega}$ changes sign there. The mean rate M of northward transfer of zonal momentum [cf. (3.6)] tends to zero as a wave approaches y_0 from either side.

The other type of latitude, $y = y_r$, say, occurs where $g(y) = 0$. The solution of (2.11) changes character as the waves cross $y = y_r$. For y near y_r, (2.11) becomes, at leading order,

$$\phi'' + g'(y_r)(y - y_r)\phi = 0. \tag{6.14}$$

The solution of this equation can readily be expressed in terms of Airy functions:

$$\phi = \mathbf{A}\operatorname{Ai}(z) + \mathbf{B}\operatorname{Bi}(z), \quad z = [-g'(y)]^{\frac{1}{3}}(y - y_r). \tag{6.15}$$

From the standard properties of Airy functions Ai and Bi are both oscillatory for $z \leqslant 0$. However, for $z > 0$, Ai is exponentially decreasing and Bi is exponentially increasing. Since the solution (6.15) is legitimate only within a distance of order $|g'(y_r)|^{-\frac{1}{3}}$, this solution must asymptotically match with the solution of (2.11) away from $y = y_r$. Thus the solution W_1 does not decay to zero and W_2 does not grow to infinity as evinced by the WKBJ method.

Now explicit expressions for the amplitude of the reflected and transmitted waves can be obtained only after the forms of U and V have been specified and the solutions W_1 and W_2 (see appendix) have been found. However, the purpose of the present paper is, as mentioned in the introduction, to indicate the general behaviour of the waves as they propagate through the layer using a full wave treatment. Indeed it is possible to envisage situations which will give rise to perfect reflexion in the vortex-sheet treatment but lead to wave amplification in a study of a finite shear layer (see example 1 below).

We shall now treat some particular cases. Without any loss of generality we take $V > 0$. Since the transformation $\omega \to -\omega$ and $k_x \to -k_x$ leaves (2.11) unchanged, we may take $\omega > 0$ and allow k_x to take positive or negative values. We also find it convenient to adopt a notation for the non-zero values of k_x where $k_y = 0$ in regions 1 and 3. Let these values be k_{0i} in region 1 and k_{3i} in region 3 ($i = 1, 2, 3$). We shall always associate $i = 1$ with the Rossby wave, $i = 2$ with the Alfvén wave branch near $k_x = \omega/(U-V)$ and $i = 3$ with the Alfvén wave branch near $k_x = \omega/(U+V)$ (see figure 1).

Example 1. *Westerly wind increasing with latitude*

Consider a westerly wind ($U > 0$) which increases steadily from zero at $y = 0$ to a maximum U_3 at $y = L$. Suppose that an Alfvén wave for which $\omega/V < k_x < k_{x3}$ is propagating in region 1 towards the shear layer. The behaviour of the wave in the layer and in region 3 will depend very much on the maximum speed U_3. For simplicity we assume that V is uniform throughout but necessarily non-zero.

If U_3 varies so slowly that $U_3 - V < \omega/k_x$ then the wave will propagate through the layer without encountering any critical latitudes to emerge in region 3 as a positive-energy propagating wave if $k_{33} > k_x > \omega/(U_3+V)$, in which case the reflexion and transmission coefficients are related by (6.6), or as an evanescent wave otherwise, in which case $|T| = 0$ and $|R| = 1$. In either case over-reflexion is not possible. However, if $U_3 - V > \omega/k_x$, the situation is different. Since U is steadily increasing, there will exist a value U_c of U such that $U_c - V = \omega/k_x$, i.e. the wave will encounter a critical latitude at $y = y_c$, say, as it moves northwards towards region 3. The form of the wave as it approaches the critical latitude, however, depends on the function $g(y)$. Since $g(y) > 0$ both in region 1 and near $y = y_c$ then either $g(y) \geqslant 0$ in $0 \leqslant y \leqslant y_c$ or $g(y)$ has an even number of zeros in this interval. If $g(y) \geqslant 0$ for all $y \leqslant y_c$ then the solution of (2.11) is oscillatory in this interval and the wave propagates all the way to the critical latitude. If, however, $g(y) < 0$ in some subintervals of $0 \leqslant y \leqslant y_c$, the solution of (2.11) is oscillatory where $g(y) \geqslant 0$ and exponential where $g(y) < 0$. Whether $g(y)$ has zeros or not the wave will eventually reach the critical latitude in the form of a propagating wave [cf. (6.7)]. Furthermore, since $\omega/(U-V) > \omega/U > \omega/(U+V)$ for

$U > 0$, the wave must pass through the latitude $y = y_0$, where the flow speed matches the zonal wave speed, before it reaches the critical latitude. At $y = y_0$ the wave changes from a positive- to a negative-energy wave in $y > y_0$. At the critical latitude $U'_c - V'_c > 0$ since $U' > 0$ and $V'_c = 0$ by hypothesis. The π-term in (6.13) is then positive and hence $|R|$ *is increased by the presence of the critical latitude*. This is an indication that wave amplification (or over-reflexion) is possible. Indeed, if $k_x > k_{32}$, the wave is evanescent in region 3, i.e. $|T| = 0$, and (6.13) reduces to

$$|R|^2 = 1 + \pi |B|^2/|k_{y1}| > 1, \tag{6.16}$$

i.e. the wave is amplified. (Note that in the vortex-sheet treatment this situation corresponds to perfect reflexion). However, if $k_x < k_{32}$, the wave will emerge in region 3 as a negative-energy propagating Alfvén wave (i.e. $|T| \neq 0$) and it is not possible, in general, to decide whether wave amplification will occur or not although it may be anticipated that wave amplification may be possible if $k_{y3}/k_{y1} \ll 1$, i.e. if the transmitted wave is a long wave. It should be pointed out here that wave amplification is due to the 'emission' of energy at the critical latitude, as is clearly shown by (6.16).

Example 2. Easterly wind increasing with latitude

Consider a velocity profile in which U decreases steadily from 0 at $y = 0$ to a minimum value U_3 at $y = L$ and also assume that V ($\neq 0$) is uniform throughout the medium. Suppose that a Rossby wave in region 1 with a prescribed k_x (i.e. $0 > k_x > k_{01}$) is incident on the layer. If $V - U_3 < \omega/(-k_x)$, the wave will propagate right through the layer without encountering a critical latitude and will emerge in region 3 as a Rossby wave or as an Alfvén–Rossby wave depending on whether $V \lessgtr V_0$. In either case wave amplification is impossible since $|R|$ and $|T|$ ($\neq 0$) are given by (6.6) with $(k_{y3}/k_{y1}) > 0$.

In the case $V - U_3 > \omega/(-k_x)$, the wave propagates until it reaches a critical latitude where $V - U_c = -\omega/k_x$, crosses this critical latitude and propagates northwards. Since $U'_c < 0$ and $V'_c = 0$, then $U'_c - V'_c < 0$ and the π-term in (6.13) is negative. The behaviour of the wave beyond this critical latitude depends on whether $U_3 \lessgtr -V + k\omega/k_x$.

If $U_3 > -V + \omega/k_x$, the wave will not encounter another critical latitude. If $k_x > k_{32}$, the wave is evanescent in region 3, i.e. $|T| = 0$, and (6.15) gives

$$|R|^2 = 1 - \pi |B|^2/|k_{y1}| < 1, \tag{6.17}$$

a result which shows that wave amplification is impossible. If, however, $k_x < k_{32}$, the wave will emerge as a positive-energy wave in region 3 and again wave amplification is impossible since $k_{y3}/k_{y1} > 0$.

If $U_3 < -V + \omega/k_x$, the wave will encounter another critical latitude where U satisfies $U_c + V = \omega/k_x$, after it has passed through the latitude $y = y_0$, where $U = \omega/k_x$, and has changed from a positive- to a negative-energy wave. As the wave crosses the second critical latitude the reflexion coefficient is *reduced further* since the π-term is also negative here. Indeed if $\mathbf{B}$ is the amplitude of the solution with a logarithmic singularity near the second critical latitude then (6.13) gives

$$|R|^2 = 1 - \pi[|B|^2 + |\mathbf{B}|^2]/|k_{y1}|, \tag{6.18}$$

since the wave will emerge as an evanescent wave in region 3 and hence $|T| = 0$. Thus the presence of a second critical latitude in an easterly wind *supplements* the first one in reducing $|R|$.

Example 3

Consider a magnetic–velocity shear in which (i) U (> 0) increases steadily from zero in region 1 to U_3 in region 3 in a such a way that $U_3 < V_1$, and (ii) V decreases steadily from V_1 in region 1 to V_3 in region 3 such that $U_3 - V_3 > 0$. Suppose that a wave for which $k_{03} > k_x > \omega/(U_1 + V_1)$ is incident on the finite shear layer from region 1. Suppose further that U_3 and V_3 are such that

$$k_x > \omega(U_3 - V_3),$$

so that the wave will encounter a critical latitude before it reaches region 3. Since $U - V$ increases steadily from negative to positive values $U_c' - V_c' > 0$ and the π-term in (6.13) is positive. It then follows that the same conclusions as were reached in example 1 apply here.

Example 4

Consider the situation when $U = 0$ everywhere, i.e. the case of a purely magnetic shear. If the incident wave is an eastward-propagating Alfvén wave, it will encounter a critical latitude at $V = \omega/k_x$ only if V decreases from V_1 to V_3, in which case $V_c' + U_c' < 0$ and the wave will emerge in region 3 as an evanescent wave. R is then given by (6.17). If, on the other hand, the incident wave is a westward-propagating Alfvén wave, it will encounter a critical latitude at $V = -\omega/k_x$ only if V increases with y. Here $U_c' - V_c' < 0$ and again the wave will be evanescent in region 3. R is given by (6.17) here also. We may therefore conclude that wave amplification is impossible in the absence of a shear flow (in the present problem).

To conclude this section, we point out that unlike the above four examples there may exist certain situations in which the critical latitudes encountered by the wave give π-terms with differing signs. Consider, for example, an easterly jet in which the velocity decreases steadily from zero to a minimum value $U_m < -V + \omega/k_x$ and thereafter increases to $U_3 > -V + \omega/k_x$. Here the wave will encounter at least three critical latitudes, the first two of which will give negative π-terms and the others positive π-terms. In such a situation no firm conclusions as to the possibility of wave amplification can be made. Similar results hold for westerly jets and for series of easterly or westerly jets.

We wish to thank a referee for pointing out an error in an earlier draft of the paper and Professor Sir James Lighthill for some helpful comments.

Appendix

Reflexion and transmission coefficients for a magnetic–velocity shear layer

Application of the boundary conditions (6.4) to the solutions (6.2) yields the equations

$$1+R = AW_1(0)+BW_2(0), \tag{A 1}$$

$$T_1 = T\exp(ik_{y3}L) = AW_1(L)+BW_2(L), \tag{A 2}$$

$$ik_{y1}(1-R) = AW_1'(0)+BW_2'(0), \tag{A 3}$$

$$ik_{y3}T_1 = AW_1'(L)+BW_2'(L). \tag{A 4}$$

In the case of discontinuities at $y = 0$, L, when the boundary conditions (6.3) apply, similar equations are obtained. Now straightforward manipulation of (A 1)–(A 4) gives

$$R = \frac{-\delta W_1'(0)-W_2'(0)+ik_{y1}[\delta W_1(0)+W_2(0)]}{\delta W_1'(0)+W_2'(0)+ik_{y1}[\delta W_1(0)+W_2(0)]}, \tag{A 5}$$

$$T_1 = \frac{\delta W_1(L)+W_2(L)}{\delta W_1(0)+W_2(0)}(1+R), \tag{A 6}$$

$$B = 2ik_{y1}/\{\delta W_1'(0)+W_2'(0)+ik_{y1}[\delta W_1(0)+W_2(0)]\}, \tag{A 7}$$

$$A = \delta B, \tag{A 8}$$

where

$$\delta = -\frac{W_2'(L)-ik_{y3}W_2(L)}{W_1'(L)-ik_{y3}W_1(L)}. \tag{A 9}$$

In deriving these equations we have taken $W_{1,2}(L)$ and $W_{1,2}'(L)$ to include the matching requirements at all critical latitudes within the layer.

Dispersion equation for ripples on a current–vortex sheet

The dispersion equation for the natural modes of a finite double (magnetic and velocity) shear layer are obtained from (A 1)–(A 4) by letting the amplitude of the incident wave be zero and setting to zero the determinant of the coefficients of the wave amplitudes. This is given by the zero of the denominator of (A 5), i.e. of the reflexion coefficient. In the case of a discontinuity this equation is obtained from (5.8). Thus

$$k_{y1}(\hat{\omega}^2-k_x^2V_1^2)+k_{y3}(\hat{\omega}^2-k_x^2V_3^2) = 0, \tag{A 10}$$

in which

$$k_{yj} = \pm i[k_x^2+\beta\hat{\omega}_j k_x/(\hat{\omega}_j^2-k_x^2V_j^2)]^{\frac{1}{2}}, \quad j = 1, 3. \tag{A 11}$$

For values of k_x for which $k_{yj}^2 < 0$ the sign of the radical in (A 11) must be chosen such that disturbances decay into the regions $y \lessgtr 0$. Similarly, for values of k_x for which $k_{yj}^2 > 0$ the sign of k_{yj} (which is now real) must be chosen so as to ensure that disturbances on either side of the sheet correspond to energy being carried away from the sheet. It is these restrictions on k_{yj} that remove the algebraic character of (A 10).

A complete discussion of the solutions of (A 10) is quite complicated (see Chandrasekhar (1961, p. 501) for a discussion of a similar problem) and is beyond the scope of this appendix. It is at present being examined in detail and we hope to report on it in the future. However, it may be indicated that it is possible to isolate certain situations in which wave amplification is possible in a *stable* current–vortex sheet.

Here we shall simply note that the magnetic field exercises a stabilizing influence. For example, if we ignore Coriolis effects we find that the sheet is stable provided that the jump in the flow speed across the sheet does not exceed $2^{\frac{1}{2}}(V_1^2+V_3^2)^{\frac{1}{2}}$. The '$\beta$-effect' is more difficult to analyse but an inspection of the asymptotic cases β small and β large indicates that it can be either a stabilizing or a destabilizing influence depending upon a variety of conditions.

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